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The rate of convergence of q-Bernstein polynomials for $0<q<1$ ☆

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Abstract

In the note, we obtain the estimates for the rate of convergence for a sequence of q-Bernstein polynomials $\{B_{n,q}(f)\}$ for $0<q<1$ by the modulus of continuity of f, and the estimates are sharp with respect to the order for Lipschitz continuous functions. We also get the exact orders of convergence for a family of functions $f(x)=x^\alpha,\ \alpha>0,\ \alpha\neq 1$, and the orders do not depend on α, unlike the classical case.

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Keywords: q-Bernstein polynomials; Rate of approximation; Modulus of continuity

1. Introduction

Let $q>0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := \begin{cases} (1-q^k)\big/(1-q), & q\neq 1 \\ k, & q=1 \end{cases}$$

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and

$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k\geqslant 1 \\ 1, & k=0 \end{cases}$$

respectively. For the integers $n,\ k,\ n\geqslant k\geqslant 0$, the q-binomial coefficients are defined by (see [3, p. 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

In 1997, Phillips proposed the following q-Bernstein polynomials $B_{n,q}(f,x)$. For each positive integer n, and $f \in C[0,1]$, we define

$$B_{n,q}(f,x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x), \tag{1.1}$$

where it is agreed that an empty product denotes 1 (see [6]). When $q=1$, $B_{n,q}(f,x)$ reduce to the well-known Bernstein polynomials $B_n(f,x)$:

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

In recent years, the q-Bernstein polynomials have attracted much interest, and a great number of interesting results related to the q-Bernstein polynomials have been obtained (see [2,5–9]). This note is concerned with the quantitative results for the rate of convergence of the q-Bernstein polynomials for $0<q<1$. For $f\in C[0,1],\ t>0$, we define the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\omega(f;t) := \sup_{\substack{|x-y|\leqslant t \\ x,y\in[0,1]}} |f(x)-f(y)|,$$

$$\omega_2(f,t) := \sup_{0<h\leqslant t}\ \sup_{x\in[0,1-2h]} |f(x+2h)-2f(x+h)+f(x)|.$$

For fixed $q\in(0,1)$, II'inskii and Ostrovska proved in [2] that for each $f\in C[0,1]$, the sequence $\{B_{n,q}(f,x)\}$ converges to $B_{\infty,q}(f,x)$ as $n\to\infty$ uniformly for $x\in[0,1]$, where

$$B_{\infty,q}(f,x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)\frac{x^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1-q^s x), & 0\leqslant x<1, \\ f(1), & x=1. \end{cases} \tag{1.2}$$

The first author of the note gave the following quantitative result for the rate of convergence of the q-Bernstein polynomials (see [9]):

$$\|B_{n,q}(f)-B_{\infty,q}(f)\| \leqslant c\ \omega_2(f,\sqrt{q^n}\ ) \tag{1.3}$$

with $\|\cdot\|$ the uniform norm, here c is an absolute constant. Note that when $f(x)=x^2$, we have (see [9]):

$$\|B_{n,q}(f)-B_{\infty,q}(f)\| = \sup_{x\in[0,1]} \frac{q^n(1-q)}{1-q^n}x(1-x) \asymp q^n \asymp \omega_2(f,\sqrt{q^n}\ ),$$

where $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant c independent of n such that $A(n) \leqslant c\, B(n)$. Hence the estimate (1.3) is sharp in the following sense: the sequence $\sqrt{q^n}$ in (1.3) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$.

In the case $q = 1$, we have (see [1, p. 308])

$$\|B_n(f) - f\| \ll \omega_2(f, n^{-1/2}) \ll \omega(f, n^{-1/2}).$$

The above estimates are both sharp in the sense of order, and for the functions $g_\alpha(x) := x^\alpha,\ \alpha > 0,\ \alpha \neq 1$, we have the following estimates:

$$\|B_n(g_\alpha) - g_\alpha\| \asymp \begin{cases} n^{-\alpha/2}, & 0 < \alpha < 2,\ \alpha \neq 1 \\ n^{-1}, & \alpha > 2 \end{cases} \asymp\ \omega_2(g_\alpha, n^{-1/2}).$$

Note that the relations $\|B_n(g_\alpha) - g_\alpha\| \asymp \omega(g_\alpha, n^{-1/2})$ hold only for $0 < \alpha < 1$. So in the case $q = 1$, the second modulus of smoothness is more appropriate to describe the degree of approximation of the Bernstein polynomials than the modulus of continuity.

What about the case $0 < q < 1$? One may think it is similar to the case $q = 1$ and conjecture that the inequality

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| \ll \omega(f, \sqrt{q^n}\,)$$

is sharp in the sense of order. However, the above conjecture is wrong. In the note, we obtain the estimates for the rate of convergence for q-Bernstein polynomials $\{B_{n,q}(f)\}$ for $0 < q < 1$ in terms of $\omega(f, \cdot)$, and the estimates are sharp with respect to the order for Lipschitz continuous functions. Our results show that in the case $0 < q < 1$, $\omega(f, \cdot)$ is more appropriate to describe the rate of convergence for q-Bernstein polynomials $\{B_{n,q}(f)\}$ than $\omega_2(f, \cdot)$ (see the following Remark 1), and this is different from that in the case $q = 1$. We also get the exact orders of approximation $\|B_{n,q}(g_\alpha) - B_{\infty,q}(g_\alpha)\|$, and unlike the classical case, the orders do not depend on the index α ($\alpha > 0,\ \alpha \neq 1$). Our main results are the following:

Theorem 1. *Let $q \in (0, 1)$, and let $f \in C[0, 1]$. Then*

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| \leqslant C_q\, \omega(f, q^n), \tag{1.4}$$

where $C_q = 2 + \frac{4 \ln \frac{1}{1-q}}{q(1-q)}$. The above estimate is sharp in the following sense of order: for each $\alpha,\ 0 < \alpha \leqslant 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class $Lip\,\alpha := \{\, f \in C[0, 1] \mid \omega(f, t) \ll t^\alpha \,\}$ such that

$$\|B_{n,q}(f_\alpha) - B_{\infty,q}(f_\alpha)\| \gg q^{\alpha n}. \tag{1.5}$$

Remark 1. Combining (1.4) and (1.5), for each $\alpha,\ 0 < \alpha \leqslant 1$, we obtain

$$\sup_{f \in Lip\,\alpha} \|B_{n,q}(f) - B_{\infty,q}(f)\| \asymp q^{\alpha n}. \tag{1.6}$$

Note that if we use the estimate (1.3), we can only get that

$$\sup_{f \in Lip\,\alpha} \|B_{n,q}(f) - B_{\infty,q}(f)\| \ll \sqrt{q^{\alpha n}}.$$

Theorem 2. *Let* $q \in (0, 1)$, *and let* $g_\alpha(x) = x^\alpha,\ \alpha > 0,\ \alpha \neq 1$. *Then*

$$\|B_{n,q}(g_\alpha) - B_{\infty,q}(g_\alpha)\| \asymp q^n. \tag{1.7}$$

2. Proofs of Theorems 1–2

In the sequel, we always assume that $q \in (0, 1),\ n \in \mathbb{N}$, and f is a continuous function on $[0, 1]$.

Proof of Theorem 1. It follows directly from (1.1) and (1.2) that $B_{n,q}(f, x)$ and $B_{\infty,q}(f, x)$ possess the end point interpolation property, in other words,

$$B_{n,q}(f, 0) = B_{\infty,q}(f, 0) = f(0), \qquad B_{n,q}(f, 1) = B_{\infty,q}(f, 1) = f(1). \tag{2.1}$$

It was proved in [2,6] that $B_{\infty,q}(f; x)$ and $B_{n,q}(f, x)$ reproduce linear functions, that is,

$$B_{n,q}(at + b, x) = B_{\infty,q}(at + b, x) = ax + b. \tag{2.2}$$

We set

$$p_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x), \qquad p_{\infty k}(q; x) := \frac{x^k}{(1-q)^k [k]!} \prod_{s=0}^{\infty} (1 - q^s x).$$

Obviously $p_{nk}(q; x) \geqslant 0,\ \ p_{\infty k}(q; x) \geqslant 0$ for all $x \in [0, 1]$. It follows from (2.2) (with $a = 0,\ b = 1$) that

$$\sum_{k=0}^{n} p_{nk}(q; x) = \sum_{k=0}^{\infty} p_{\infty k}(q; x) = 1. \tag{2.3}$$

Hence for all $x \in (0, 1)$, by the definitions of $B_{n,q}(f, x)$ and $B_{\infty,q}(f, x)$, and by (2.3) we know that

$$\begin{aligned}
&|B_{n,q}(f, x) - B_{\infty,q}(f, x)| \\
&\quad = \left| \sum_{k=0}^{n} f([k]/[n]) p_{nk}(q; x) - \sum_{k=0}^{\infty} f(1 - q^k) p_{\infty k}(q; x) \right| \\
&\quad = \left| \sum_{k=0}^{n} (f([k]/[n]) - f(1)) p_{nk}(q; x) - \sum_{k=0}^{\infty} (f(1 - q^k) - f(1)) p_{\infty k}(q; x) \right| \\
&\quad \leqslant \sum_{k=1}^{n} |f([k]/[n]) - f(1 - q^k)| p_{nk}(q; x) \\
&\qquad + \sum_{k=0}^{n} |f(1 - q^k) - f(1)|\, |p_{nk}(q; x) - p_{\infty k}(q; x)| \\
&\qquad + \sum_{k=n+1}^{\infty} |f(1 - q^k) - f(1)| p_{\infty k}(q; x) =: I_1 + I_2 + I_3.
\end{aligned} \tag{2.4}$$

First we estimate $I_1,\ I_3$. Since

$$0 \leqslant \frac{[k]}{[n]} - (1-q^k) = \frac{1-q^k}{1-q^n} - (1-q^k) = \frac{q^n(1-q^k)}{1-q^n} \leqslant q^n,$$

$$0 \leqslant 1-(1-q^k) = q^k \leqslant q^n \quad (k \geqslant n+1),$$

we get

$$I_1 \leqslant \omega(f,q^n) \sum_{k=0}^{n} p_{nk}(q;x) = \omega(f,q^n) \tag{2.5}$$

and

$$I_3 \leqslant \omega(f,q^n) \sum_{k=n+1}^{\infty} p_{\infty k}(q;x) \leqslant \omega(f,q^n). \tag{2.6}$$

Now we estimate I_2. Using the property of modulus of continuity (see [4, p. 20])

$$\omega(f,\lambda t) \leqslant (1+\lambda)\omega(f,t), \quad \lambda > 0,$$

we get

$$\begin{aligned} I_2 &\leqslant \sum_{k=0}^{n} \omega(f,q^k)|p_{nk}(q;x) - p_{\infty k}(q;x)| \\ &\leqslant \sum_{k=0}^{n} \omega(f,q^n)(1+q^{k-n})|p_{nk}(q;x) - p_{\infty k}(q;x)| \\ &\leqslant 2\omega(f,q^n)\frac{1}{q^n}\sum_{k=0}^{n} q^k|p_{nk}(q;x) - p_{\infty k}(q;x)| =: \frac{2}{q^n}\,\omega(f,q^n)\cdot J, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} J &:= \sum_{k=0}^{n} q^k \left| \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x) - \frac{x^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1-q^s x) \right| \\ &= \sum_{k=0}^{n} q^k \left| \begin{bmatrix} n \\ k \end{bmatrix} x^k \left(\prod_{s=0}^{n-k-1}(1-q^s x) - \prod_{s=0}^{\infty}(1-q^s x) \right) \right. \\ &\quad \left. + x^k \prod_{s=0}^{\infty}(1-q^s x)\left(\begin{bmatrix} n \\ k \end{bmatrix} - \frac{1}{(1-q)^k[k]!} \right) \right| \\ &\leqslant \sum_{k=0}^{n} q^k p_{nk}(q;x) \left| 1 - \prod_{s=n-k}^{\infty}(1-q^s x) \right| \\ &\quad + \sum_{k=0}^{n} q^k p_{\infty k}(q;x) \left| \prod_{s=n-k+1}^{n}(1-q^s) - 1 \right|. \end{aligned}$$

Assume for a moment that the inequality

$$1-\prod_{s=j}^{\infty}(1-q^s)\leqslant \frac{q^j}{q(1-q)}\ln\frac{1}{1-q}\quad (j=1,2,\ldots) \tag{2.8}$$

holds. Then we obtain

$$\left|1-\prod_{s=n-k}^{\infty}(1-q^s x)\right|\leqslant 1-\prod_{s=n-k}^{\infty}(1-q^s)\leqslant \frac{q^{n-k}}{q(1-q)}\ln\frac{1}{1-q},$$

and

$$\left|1-\prod_{s=n-k+1}^{n}(1-q^s)\right| \leqslant 1-\prod_{s=n-k+1}^{\infty}(1-q^s)\leqslant \frac{q^{n-k+1}}{q(1-q)}\ln\frac{1}{1-q}\leqslant \frac{q^{n-k}}{q(1-q)}\ln\frac{1}{1-q}.$$

Hence

$$J\leqslant \frac{q^n}{q(1-q)}\ln\frac{1}{1-q}\left(\sum_{k=0}^{n}p_{nk}(q;x)+\sum_{k=0}^{n}p_{\infty k}(q;x)\right)\leqslant \frac{2q^n}{q(1-q)}\ln\frac{1}{1-q}. \tag{2.9}$$

From (2.1), (2.4)–(2.7), and (2.9), we conclude that

$$\|B_{n,q}(f)-B_{\infty,q}(f)\|\leqslant C_q\,\omega(f,q^n),$$

where $C_q=2+\frac{4\ln\frac{1}{1-q}}{q(1-q)}$.

Now we return to the inequality (2.8) left open above. We note that the inequality $1-\exp(-x)\leqslant x$ holds for each $x\in[0,\infty)$. Then

$$1-\prod_{s=j}^{\infty}(1-q^s)=1-\exp\left(-\sum_{s=j}^{\infty}\ln\frac{1}{1-q^s}\right)\leqslant \sum_{s=j}^{\infty}\ln\frac{1}{1-q^s}.$$

From the monotonicity of the function $\frac{1}{x}\ln\frac{1}{1-x}=\sum_{n=0}^{\infty}\frac{x^n}{n+1},\ x\in[0,1)$ we know that for all $x\in(0,q]$,

$$\ln\frac{1}{1-x}\leqslant \frac{x}{q}\ln\frac{1}{1-q}.$$

Hence

$$1-\prod_{s=j}^{\infty}(1-q^s)\leqslant \sum_{s=j}^{\infty}q^{s-1}\ln\frac{1}{1-q}=\frac{q^j}{q(1-q)}\ln\frac{1}{1-q}.$$

At last we show that the estimate (1.4) is sharp. For each $\alpha,\ 0 < \alpha \leqslant 1$, suppose that $f_\alpha(x)$ is a continuous function which is equal to zero in $[0, 1-q]$ and $[1-q^2, 1]$, equal to $(x-(1-q))^\alpha$ in $[1-q, 1-q+q(1-q)/2]$, and linear in the rest of [0, 1]. Then

$$\omega(f_\alpha, t) \asymp t^\alpha,$$

and

$$\|B_{n,q}(f_\alpha) - B_{\infty,q}(f_\alpha)\| \asymp q^{\alpha n}\|p_{n1}(q;\cdot)\| \asymp q^{\alpha n}.$$

The proof of Theorem 1 is complete. □

Denote by $\omega(f,t)_{[1-q,1]}$ the modulus of continuity on the interval $[1-q, 1]$, that is,

$$\omega(f;t)_{[1-q,1]} = \sup_{\substack{|x-y|\leqslant t \\ x,y\in[1-q,1]}} |f(x)-f(y)|.$$

Then we have

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| \ll q^n\ \|f\| + \omega(f,q^n)_{[1-q,1]}. \tag{2.10}$$

In fact, if we take continuous $\tilde{f} = f$ on $[1-q,1]\bigcup\{0\}$ and linear on $[0, 1-q]$, we get $B_{n,q}(f) = B_{n,q}(\tilde{f})$ and $B_{\infty,q}(f) = B_{\infty,q}(\tilde{f})$. At the same time $\omega(\tilde{f},t) \ll \|f\|\,t + \omega(f,t)_{[1-q,1]}$. By (1.4) we get (2.10).

Remark 2. From the proof of Theorem 1, we know that the rate of convergence for q-Bernstein polynomials $B_{n,q}(f,x)$ depends only on the smoothness of the function $f(x)$ at the points $1-q^k,\ k = 1, 2, \cdots$ (from the right), and at $x = 1$.

Proof of Theorem 2. Since $\omega(x^\alpha, t)_{[1-q,1]} \asymp t$ holds for each $\alpha > 0$, $\alpha \neq 1$, by (2.10) we obtain that

$$\|B_{n,q}(t^\alpha) - B_{\infty,q}(t^\alpha)\| \ll q^n.$$

Now we prove the lower estimates of $\|B_{n,q}(t^\alpha) - B_{\infty,q}(t^\alpha)\|$ for all $\alpha > 0,\ \alpha \neq 1$. From the formulas (7.61) and (7.62) in [7, p. 270], we know if f is a convex (concave) function, then the sequence $\{B_{n,q}(f,x)\}$ is decreasing (increasing) for each $x \in [0, 1]$, and also

$$|B_{n,q}(f,x) - B_{n+1,q}(f,x)| = \sum_{k=1}^{n} |a_{nk}(f)|\ p_{n+1\,k}(q;x),$$

where

$$a_{nk}(f) = \frac{[n+1-k]}{[n+1]} f\left(\frac{[k]}{[n]}\right) + q^{n+1-k}\frac{[k]}{[n+1]} f\left(\frac{[k-1]}{[n]}\right) - f\left(\frac{[k]}{[n+1]}\right). \tag{2.11}$$

Since the functions x^α are concave for all $0 < \alpha < 1$ and convex for all $\alpha > 1$, then for all $\alpha > 0,\ \alpha \neq 1$ we get

$$|B_{n,q}(t^\alpha,x) - B_{\infty,q}(t^\alpha,x)| \geqslant |B_{n,q}(t^\alpha,x) - B_{n+1,q}(t^\alpha,x)| \geqslant |a_{n1}(t^\alpha)|\ p_{n+1\,1}(q;x). \tag{2.12}$$

It follows from (2.11) that

$$\begin{aligned} &q^{-n}|a_{n1}(x^{\alpha})| \\ &\quad = q^{-n}\left|\frac{[n]}{[n+1]}\left(\frac{[1]}{[n]}\right)^{\alpha} - \left(\frac{[1]}{[n+1]}\right)^{\alpha}\right| = \frac{q^{-n}}{[n+1]}\left|\,[n]^{1-\alpha} - [n+1]^{1-\alpha}\right| \\ &\quad = \frac{q^{-n}}{[n+1]}\left|\,(1-\alpha)\,\xi^{-\alpha}\,([n]-[n+1])\right| \quad (\text{where } \xi \in ([n],[n+1])) \\ &\quad = \frac{1-q}{1-q^{n+1}}\left|(1-\alpha)\xi^{-\alpha}\right| \to (1-q)^{1+\alpha}\,|1-\alpha| \quad \text{as } n \to \infty, \end{aligned} \tag{2.13}$$

where in the last equality, we use the fact that $\xi^{-1} \to 1-q$ as $n \to \infty$. From (2.12) and (2.13) we conclude that

$$\|B_{n,q}(t^{\alpha}) - B_{\infty,q}(t^{\alpha})\| \geqslant |a_{n1}(x^{\alpha})|\ \|\ p_{n+1\,1}(q;\cdot)\| \gg q^{n}.$$

The proof of Theorem 2 is finished. □

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